

# Post-Newtonian Approximation of Teleparallel Gravity Coupled with a Scalar Field

Zu-Cheng Chen,<sup>1,\*</sup> You Wu,<sup>2,†</sup> and Hao Wei<sup>1,‡</sup>

<sup>1</sup>*School of Physics, Beijing Institute of Technology, Beijing 100081, China*

<sup>2</sup>*Department of Physics, Renmin University of China, Beijing 100872, China*

## ABSTRACT

We use the parameterized post-Newtonian (PPN) formalism to explore the weak field approximation of teleparallel gravity non-minimally coupling to a scalar field  $\phi$ , with arbitrary coupling function  $\omega(\phi)$  and potential  $V(\phi)$ . We find that all the PPN parameters are identical to general relativity (GR), which makes this class of theories compatible with the Solar System experiments. This feature also makes the theories quite different from the scalar-tensor theories, which might be subject to stringent constraints on the parameter space, or need some screening mechanisms to pass the Solar System experimental constraints.

PACS numbers: 04.25.Nx, 04.50.Kd, 95.36.+x, 04.80.-y

---

\* Email address: [bingining@gmail.com](mailto:bingining@gmail.com)

† Email address: [5u@ruc.edu.cn](mailto:5u@ruc.edu.cn)

‡ Corresponding author; Email address: [haowei@bit.edu.cn](mailto:haowei@bit.edu.cn)

## I. INTRODUCTION

Since the late-time acceleration of our universe has been confirmed by various observations [1–9], in the literature there are much efforts to explain this surprising phenomenon. Although it is straightforward to introduce a cosmological constant [10–16] to account for the acceleration, it also gives rise to the so-called fine-tuning [10] and cosmic coincidence [17] problems. Other kinds of dark energy (DE) [18–20], e.g. quintessence [21–23], phantom [24, 25],  $k$ -essence [26–29], and tachyon [30–34], work as well, but one also needs to figure out why it is homogeneous [35] and why it has recently achieved dominance [13–15, 18]. On the other hand, because the energy scale of the field potential is very low, it is difficult to construct viable scalar field models in the framework of particle physics. Instead of assuming the existence of a mysterious DE with exotic properties, an alternative approach is to modify Einstein’s general relativity (GR) on the cosmological scales while GR can be restored on small scales. In the literature, such approaches are usually called modified gravity theories [36, 37]. In particular, the scalar-tensor theories [38–42] which introduce an extra degree of freedom, namely a scalar field  $\phi$ , coupling to the gravitational sector (the Ricci scalar  $R$ ), might be one of the most natural alternatives to GR, since such a scalar field generically could arise in the attempts to quantize gravity (e.g. string theory). Scalar-tensor theories can not only describe the deviation from GR to have the desired cosmological dynamics on large scales [43–45], but they also respect most of the GR’s symmetries, e.g. the local Lorentz invariance.

On the other hand, torsion tensor can naturally arise when one studies the gauge theories which try to quantize gravitational field and unify it with other fundamental interactions. In fact, spin and torsion can be formulated naturally and elegantly in such gauge formulations of gravity [46, 47]. By introducing the curvatureless Weitzenböck connection [48] instead of the torsionless Levi-Civita connection used in GR, the so-called Teleparallel Equivalent of General Relativity (TEGR), or also known as teleparallel gravity, can be formulated, which naturally arises within the framework of the gauge theory of the spacetime translation group. Teleparallel gravity uses the vierbein field as the basic dynamical quantity instead of metric in GR, and attributes gravitation to the torsion tensor. After it was originally proposed by Einstein in 1920s [49–52], teleparallel gravity has been extensively studied in the literature (see e.g. [53–57]). As is well known, the Friedmann-Robertson-Walker (FRW) universe in the framework of teleparallel gravity is completely equivalent to a matter-dominated universe in the framework of GR, and hence cannot be accelerated. In the literature, there are two ways out. In analogy to the well-known  $f(R)$  theory, the first approach is to generalize teleparallel gravity to the so-called  $f(T)$  theory [58, 59]. The second approach is to directly add DE into teleparallel gravity. Of course, the simplest candidate of DE is quintessence characterized by a canonical scalar field. Inspired by the well-known scalar-tensor theories, it is reasonable to introduce a non-minimal coupling between the scalar field and the torsion scalar  $T$ . The so-called teleparallel dark energy [60–67], in which the canonical scalar field (quintessence) is coupled with the gravitation, has been shown that it can drive the cosmic acceleration even when the potential of the scalar field vanishes [65, 66]. Note that in e.g. [60–67] the coupling is chosen to be a particular form. Later, the teleparallel dark energy model has been generalized in various directions. For instance, the so-called tachyonic teleparallel dark energy model, in which a non-canonical scalar field (tachyon field) is coupled with gravitation, has been shown that the effective equation-of-state parameter (EoS) of DE can cross the phantom divide, and the cosmological coincidence problem could be alleviated [68–71]. Noether symmetry has been studied [72, 99] in the teleparallel dark energy model, in which the coupling constant is extended to be a general coupling function. It is claimed that the effective EoS can cross the phantom divide if the coupling function and the potential of the scalar field are of power-law forms.

No matter how successful an alternative theory to GR is on the cosmological scales, it should also have the appropriate Newtonian and post-Newtonian approximations in order to pass the local tests in Solar System. As is well known, a natural framework to test the weak field limit of a gravity theory is given by the parameterized post-Newtonian (PPN) formalism (see e.g. [73]). In fact, modified gravity theories are usually subject to much severer constraints from the Solar System experiments than the ones from cosmological observations. For instance, the parameter of the prototypical Brans-Dicke theory  $\omega_{\text{BD}}$  [74] was constrained to  $\omega_{\text{BD}} > 181.65$  at  $2\sigma$  confidence level by using Planck data of the cosmic microwave background (CMB) combined with the baryon acoustic oscillation (BAO) data in [75], and  $\omega_{\text{BD}} > 40000$  at  $2\sigma$  confidence level by using the tracking data obtained from the Cassini mission [76]. On the other hand, some types of modified gravity theories are even claimed to be incompatible with the local tests in Solar System [77–79], and hence cannot be viable candidates to explain the cosmic acceleration. In the

more general scalar-tensor theories and  $f(R)$  theories, the well-known Chameleon mechanism is invoked to screen the fifth force [80–83], and hence they have no significant deviation from GR on small scale, while they can still drive the acceleration of the universe on cosmological scale. Similarly, the Vainshtein mechanism [84–86] and the Symmetron mechanism [87, 88] are also extensively invoked in other types of modified gravity theories to pass the local tests in Solar System.

Motivated by the above discussions, it is necessary and worth to explore the weak field behaviors of modified gravities. Recently, the PPN parameters for the teleparallel dark energy model have been explicitly calculated in [67], and it is claimed that the potential of the scalar field has no effect on PPN parameters and hence this model can be compatible with the local tests in Solar System. Note that in [67] the coupling is chosen to be a particular form. In the present work, we try to generalize the work of [67] and explore the weak field approximation of teleparallel gravity non-minimally coupling to a scalar field  $\phi$  with arbitrary coupling function  $\omega(\phi)$  and potential  $V(\phi)$ , by explicitly calculating the corresponding PPN parameters. This paper is organized as follows. We give a brief review of teleparallel gravity in Sec. II. Next, we present the action functional for the teleparallel gravity coupled with a scalar field and derive the corresponding field equations in Sec. III. We then expand the field equations to sufficient orders and solve the perturbations to obtain the post-Newtonian approximation in Sec. IV. Finally, some concluding remarks are given in Sec. V.

## II. TELEPARALLEL GRAVITY

Here we give a brief review of teleparallel gravity. Teleparallel gravity uses a vierbein field  $e_a = e_a^\mu \partial_\mu$  as dynamical quantity, with Latin indices  $a, b, \dots = 0, \dots, 3$ , and  $i, j, \dots = 1, \dots, 3$ , Greek indices  $\mu, \nu, \dots = 0, \dots, 3$ , and  $\partial_\mu$  coordinate bases. We also note that the Einstein summation notation for the indices is used throughout this work. The vierbein is an orthonormal basis for the tangent space at each point  $x^\mu$  of the manifold, namely  $e_a \cdot e_b = \eta_{ab}$ , with  $\eta_{ab} = \text{diag}(-1, 1, 1, 1)$ . Then the metric tensor can be expressed in the dual vierbein  $e^a_\mu$  as

$$g_{\mu\nu}(x) = \eta_{ab} e^a_\mu(x) e^b_\nu(x). \quad (2.1)$$

Rather than using the torsionless Levi-Civita connection in GR, teleparallel gravity uses the Weitzenböck connection  $\Gamma^\lambda_{\mu\nu}$  [48], which is defined by

$$\Gamma^\lambda_{\mu\nu} = e_a^\lambda \partial_\mu e^a_\nu. \quad (2.2)$$

Note that the lower indices  $\mu$  and  $\nu$  are not symmetric in general, thus the torsion tensor (will be defined below) is non-vanishing in the teleparallel spacetime. The Weitzenböck torsion tensor is defined by

$$T^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu} - \Gamma^\lambda_{\mu\nu} = e_a^\lambda (\partial_\nu e^a_\mu - \partial_\mu e^a_\nu). \quad (2.3)$$

In teleparallel gravity, the gravitational action is given by the torsion scalar instead of the Ricci scalar in GR. The torsion scalar is basically the square of the Weitzenböck torsion tensor, and reads

$$T = S^\rho_{\mu\nu} T^\mu{}_\rho{}^{\nu} = \frac{1}{4} T^\rho_{\mu\nu} T^\mu{}_\rho{}^{\nu} + \frac{1}{2} T^\rho_{\mu\nu} T^{\nu\mu}{}_\rho - T^\rho_{\mu\rho} T^{\nu\mu}{}_\nu, \quad (2.4)$$

with the super-potential tensor  $S^\rho_{\mu\nu}$  defined by

$$S^\rho_{\mu\nu} = \frac{1}{4} (T^\rho_{\mu\nu} - T_{\mu\nu}{}^\rho + T_{\nu\mu}{}^\rho) + \frac{1}{2} \delta^\rho_\mu T^\sigma_{\nu\sigma} - \frac{1}{2} \delta^\rho_\nu T^\sigma_{\mu\sigma}. \quad (2.5)$$

The gravitational field is driven by the torsion scalar  $T$ , and the action reads

$$S = \frac{1}{2\kappa^2} \int e T d^4x + S_m[e_a^\mu, \chi_m], \quad (2.6)$$

where  $e = \det(e_a^\mu) = \sqrt{-g}$  and  $\kappa^2 = 8\pi G_N$ , with  $g$  the determinant of the metric  $g_{\mu\nu}$  and  $G_N$  the Newtonian constant. Note that we have used the units in which the speed of light  $c = 1$ , and the reduced Planck constant  $\hbar = 1$ .  $S_m[e_a^\mu, \chi_m]$  is the matter part of the action, and  $\chi_m$  denotes all matter fields collectively.

### III. TELEPARALLEL GRAVITY WITH A SCALAR

We will study the theories of teleparallel gravity coupled with a scalar in which gravity is described by a dynamical scalar  $\phi$  in addition to the vierbein  $e_a^\mu$ . Without loss of generality, we consider the Brans-Dicke-like theories, whose actions are given by

$$S = \frac{1}{2\kappa^2} \int dx^4 e \left[ \phi T - \frac{\omega(\phi)}{\phi} (\partial\phi)^2 - 2\kappa^2 V(\phi) \right] + S_m[e_a^\mu, \chi_m]. \quad (3.1)$$

where the coupling function  $\omega(\phi)$  and the potential  $V(\phi)$  are two arbitrary functions of  $\phi$ . At first glance, one might consider that this action is not so general. In fact, we can make it more familiar. Introducing a new scalar  $\tilde{\phi}$  according to  $(\partial\tilde{\phi})^2 = -\omega(\phi)(\partial\phi)^2/(\kappa^2\phi)$ , Eq. (3.1) can be recast as

$$\tilde{S} = \int dx^4 e \left[ \tilde{\omega}(\tilde{\phi}) \frac{T}{2\kappa^2} + \frac{1}{2} (\partial\tilde{\phi})^2 - \tilde{V}(\tilde{\phi}) \right] + S_m[e_a^\mu, \chi_m]. \quad (3.2)$$

Obviously, if  $\tilde{\omega}(\tilde{\phi}) = 1 + \xi\kappa^2\tilde{\phi}^2$ , Eq. (3.2) reduces to the action considered in [67], namely

$$\tilde{S} = \int dx^4 e \left\{ \frac{T}{2\kappa^2} + \frac{1}{2} [(\partial\tilde{\phi})^2 + \xi T\tilde{\phi}^2] - \tilde{V}(\tilde{\phi}) \right\} + S_m[e_a^\mu, \chi_m]. \quad (3.3)$$

So, the action (3.1) is general enough in fact (see Sec. V for further discussion). Actually, the action (3.2) has been considered in e.g. [72, 89, 97] as a generalization of the action (3.3). We stress that the action (3.2) has richer structure and more physical implication than the action (3.3), thus justifying the worth of our work. For instance, it is claimed in [89] that the action (3.2) might admit the scaling attractors to alleviate the cosmological coincidence problem, while no scaling attractor has been found by performing dynamical analysis of the action (3.3) (see e.g. [62, 63]). On the other hand, using Lagrange multiplier,  $f(T)$  gravity can be recast in a form like the action (3.1) with  $\omega(\phi) = 0$  (see e.g. [98]). Despite the action (3.1) is more general than the action (3.3) because it can also encompass  $f(T)$  theory when  $\omega(\phi) = 0$ , we will not consider the case of  $\omega(\phi) = 0$  in this work, since then  $\phi$  will not be a dynamical quantity.

The variation of the action (3.1) with respect to the scalar field  $\phi$  yields

$$T - \left( \frac{\omega'}{\phi} + \frac{\omega}{\phi^2} \right) (\partial\phi)^2 + \frac{2}{\phi} \partial_\mu \omega \partial^\mu \phi + 2 \frac{\omega}{\phi} \square \phi - 2\kappa^2 V' = 0, \quad (3.4)$$

where a prime denotes a derivative with respect to  $\phi$ , and  $\square = g^{\mu\nu} \nabla_\mu \nabla_\nu$  is the d'Alembert operator, with  $\nabla_\mu$  the covariant derivative associated with the Levi-Civita connection. The variation of the action (3.1) with respect to the dual vierbein  $e^\alpha_\nu$  yields

$$\phi e^{-1} \partial_\sigma (e S_a^{\sigma\nu}) + S_a^{\sigma\nu} \partial_\sigma \phi + \phi \left( T^\sigma_{\alpha a} S_\sigma^{\nu\alpha} + \frac{T}{4} e_a^\nu \right) - \frac{1}{4} e_a^\nu \left[ \frac{\omega}{\phi} (\partial\phi)^2 + 2\kappa^2 V \right] = \frac{\kappa^2}{2} \mathcal{T}_a^\nu, \quad (3.5)$$

where  $\mathcal{T}_a^\nu \equiv -e^{-1} \delta S_m / \delta e^\alpha_\nu$  is the energy-momentum of matter. Let us bring Eq. (3.5) to a more suitable form for our purpose. Multiplying each side of Eq. (3.5) by the dual vierbein  $e^\alpha_\mu$ , we get

$$\phi e^{-1} e^\alpha_\mu \partial_\sigma (e S_a^{\sigma\nu}) + S_\mu^{\sigma\nu} \partial_\sigma \phi + \phi \left( T^\sigma_{\alpha\mu} S_\sigma^{\nu\alpha} + \frac{T}{4} \delta_\mu^\nu \right) - \frac{1}{4} \delta_\mu^\nu \left[ \frac{\omega}{\phi} (\partial\phi)^2 + 2\kappa^2 V \right] = \frac{\kappa^2}{2} \mathcal{T}_\mu^\nu, \quad (3.6)$$

where we have used the vierbein (or dual vierbein) to switch from Latin to Greek indices and back, for example  $\mathcal{T}_\mu^\nu = e^\alpha_\mu \mathcal{T}_a^\nu$ . Taking the trace of Eq. (3.6) leads to

$$\phi e^{-1} e^\alpha_\rho \partial_\sigma (e S_a^{\sigma\rho}) + S_\rho^{\sigma\rho} \partial_\sigma \phi - \left[ \frac{\omega}{\phi} (\partial\phi)^2 + 2\kappa^2 V \right] = \frac{\kappa^2}{2} \mathcal{T}, \quad (3.7)$$

with  $\mathcal{T} = \mathcal{T}_\mu^\mu$ . Multiplying Eq. (3.7) by  $(-\delta_\mu^\nu/2)$ , then adding Eq. (3.6), we get

$$\begin{aligned} & \phi e^{-1} e^\alpha_\mu \partial_\sigma (e S_a^{\sigma\nu}) - \frac{1}{2} \delta_\mu^\nu \phi e^{-1} e^\alpha_\rho \partial_\sigma (e S_a^{\sigma\rho}) + S_\mu^{\sigma\nu} \partial_\sigma \phi - \frac{1}{2} \delta_\mu^\nu S_\rho^{\sigma\rho} \partial_\sigma \phi \\ & + \phi \left( T^\sigma_{\alpha\mu} S_\sigma^{\nu\alpha} + \frac{T}{4} \delta_\mu^\nu \right) + \frac{1}{4} \delta_\mu^\nu \left[ \frac{\omega}{\phi} (\partial\phi)^2 + 2\kappa^2 V \right] = \frac{\kappa^2}{2} \left( \mathcal{T}_\mu^\nu - \frac{1}{2} \delta_\mu^\nu \mathcal{T} \right). \end{aligned} \quad (3.8)$$

The gravitational fields are truly governed by the field equations (3.4) and (3.8). We will expand these two equations in the post-Newtonian approximation in the following section.

#### IV. POST-NEWTONIAN APPROXIMATION

The post-Newtonian approximation of GR on the behavior of hydrodynamic systems has been systematically investigated in e.g. [90]. In analogy to [90], we assume that the gravitating source matter is contributed by a perfect fluid which obeys the post-Newtonian hydrodynamics. We will use the PPN formalism to expand the field equations (3.4) and (3.8) perturbatively by assigning appropriate orders of magnitude to all dynamical variables appearing in the field equations. The resulting perturbation equations can then be subsequently solved order by order.

##### A. General framework

Conventionally, the velocity of the source matter  $|\vec{v}|$  characterize the smallness of the system. So, we will perturbatively expand all dynamical quantities in orders of  $\mathcal{O}(n) \sim |\vec{v}|^n$ . We will firstly find out the perturbations for the vierbein following [67], and then expand the energy-momentum tensor to sufficient orders. Finally, the perturbations of all functions of  $\phi$  are obtained by using Taylor expansion.

For the gravitational sector, we expand the dual vierbein fields around the flat background as

$$e^a{}_\mu = \delta^a{}_\mu + B^a{}_\mu = \delta^a{}_\mu + {}^{(1)}B^a{}_\mu + {}^{(2)}B^a{}_\mu + {}^{(3)}B^a{}_\mu + {}^{(4)}B^a{}_\mu + \mathcal{O}(5), \quad (4.1)$$

where each term  ${}^{(n)}B^a{}_\mu$  is of order  $\mathcal{O}(n)$ . By using Eq. (2.1), this decomposition gives the usual metric as an expansion around the flat Minkowski background,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} = \eta_{\mu\nu} + {}^{(1)}h_{\mu\nu} + {}^{(2)}h_{\mu\nu} + {}^{(3)}h_{\mu\nu} + {}^{(4)}h_{\mu\nu} + \mathcal{O}(5), \quad (4.2)$$

where  $\eta_{\mu\nu}$  is the Minkowski metric and each symmetric term  ${}^{(n)}h_{\mu\nu}$  is of order  $\mathcal{O}(n)$ . For our purpose, it is sufficient to expand the metric up to the order of  $\mathcal{O}(4)$ . A detailed analysis (see e.g. [73]) shows that  ${}^{(1)}h_{\mu\nu} = 0$ , which corresponds to  ${}^{(1)}B^a{}_\mu = 0$  (nb. Eq. (2.1)), and the only non-vanishing components of the metric perturbations are

$${}^{(2)}h_{00}, \quad {}^{(2)}h_{ij}, \quad {}^{(3)}h_{0i}, \quad {}^{(4)}h_{00}. \quad (4.3)$$

Following [67], we denote  $B^\nu{}_\mu = \delta_a{}^\nu B^a{}_\mu$ , or equivalently  $B^a{}_\mu = \delta^a{}_\nu B^\nu{}_\mu$ , with  $\delta^a{}_\mu$  defined by  $\eta_{\mu\nu} = \eta_{ab}\delta^a{}_\mu\delta^b{}_\nu$ . We now can raise and lower the spacetime indices of the perturbations of vierbein (or dual vierbein) by the Minkowski metric  $\eta_{\mu\nu}$ ,

$$B_{\mu\nu} = \eta_{\mu\rho}B^\rho{}_\nu. \quad (4.4)$$

As a result,  $B_{\mu\nu}$  is symmetric, and the non-vanishing components are

$${}^{(2)}B_{00}, \quad {}^{(2)}B_{ij}, \quad {}^{(3)}B_{0i}, \quad {}^{(4)}B_{00}. \quad (4.5)$$

In addition,  ${}^{(2)}B_{ij}$  is diagonal [67]. For convenience, we introduce a time-independent function  $A$ , such that  ${}^{(2)}B_{ij} = A\delta_{ij}$ . We also give the relations between the metric perturbations and the vierbein perturbations [67],

$${}^{(2)}h_{00} = 2 {}^{(2)}B_{00}, \quad (4.6a)$$

$${}^{(2)}h_{ij} = 2 {}^{(2)}B_{ij}, \quad (4.6b)$$

$${}^{(3)}h_{0i} = 2 {}^{(3)}B_{0i}, \quad (4.6c)$$

$${}^{(4)}h_{00} = 2 {}^{(4)}B_{00} - {}^{(2)}B_{00} {}^{(2)}B_{00}. \quad (4.6d)$$

From the definitions, we see that  $T^\rho_{\mu\nu}$  and  $S^\rho_{\mu\nu}$  are at least  $\mathcal{O}(2)$  quantities, and the torsion scalar  $T$  is an at least  $\mathcal{O}(4)$  quantity.

The energy-momentum tensor of a perfect fluid takes the form

$$\mathcal{T}^{\mu\nu} = (\rho + \rho\Pi + p)u^\mu u^\nu + pg^{\mu\nu}, \quad (4.7)$$

where  $\rho$ ,  $\Pi$ ,  $p$  and  $u^\mu$  are the energy density, the specific internal energy, the pressure, and the four-velocity of the fluid, respectively. Note that the velocity of the source matter is given by  $v^i = u^i/u^0$ . We assign the velocity orders  $\rho \sim \Pi \sim \mathcal{O}(2)$ , and  $p \sim \mathcal{O}(4)$  by considering their orders of magnitude in the Solar System [73]. Then we get the perturbations of energy-momentum tensor in Eq. (4.7) as

$$\mathcal{T}_0^0 = -\rho(1 + v^2 + \Pi) + \mathcal{O}(6), \quad (4.8a)$$

$$\mathcal{T}_0^i = -\rho v^i + \mathcal{O}(5), \quad (4.8b)$$

$$\mathcal{T}_i^j = \rho v_i v^j + p\delta_i^j + \mathcal{O}(6). \quad (4.8c)$$

We also note that  $\mathcal{T} = g_{\mu\nu}\mathcal{T}^{\mu\nu} = -\rho - \rho\Pi + 3p$ . In addition, we assume the gravitational field is quasi-static, so the time derivative  $\partial_0 = \partial/\partial t$  of the vierbein or other fields are weighted with an additional velocity order  $\mathcal{O}(1)$ .

For the scalar field  $\phi$ , we expand it around its cosmological background value  $\phi_0$ ,

$$\phi = \phi_0 + \psi = \phi_0 + \psi^{(2)} + \psi^{(4)} + \mathcal{O}(6), \quad (4.9)$$

where we assume  $\phi_0$  to be of order  $\mathcal{O}(0)$  and the perturbations  $\psi^{(n)}$  are of order  $\mathcal{O}(n)$  as usual. We also need to expand the functions  $\omega(\phi)$  and  $V(\phi)$  around  $\phi_0$ . To this end, we expand them using Taylor expansion to sufficient orders,

$$\omega = \omega_0 + \omega_1\psi + \mathcal{O}(\psi^2), \quad (4.10a)$$

$$V = V_0 + V_1\psi + V_2\psi^2 + V_3\psi^3 + \mathcal{O}(\psi^4), \quad (4.10b)$$

with  $\omega_0 = \omega(\phi_0)$ ,  $\omega_1 = \omega'(\phi_0)$ ,  $V_0 = V(\phi_0)$ ,  $V_1 = V'(\phi_0)$ ,  $V_2 = \frac{1}{2}V''(\phi_0)$ , and  $V_3 = \frac{1}{6}V'''(\phi_0)$ . We assume all these expansion coefficients to be of order  $\mathcal{O}(0)$ . We also give the expansion of  $\omega'$  and  $V'$  for further convenience,

$$\omega' = \omega_1 + \mathcal{O}(\psi), \quad (4.10c)$$

$$V' = V_1 + 2V_2\psi + 3V_3\psi^2 + \mathcal{O}(\psi^3). \quad (4.10d)$$

## B. Solving the perturbed equations

Here we will solve the perturbed equations order by order. We refer to Appendix A for a detailed computation of the corresponding quantities up to the appropriate orders. In the followings, we just give the results directly.

Expanding Eqs. (3.8) and (3.4) to  $\mathcal{O}(0)$  simply gives the solutions  $V_0 = V_1 = 0$ . We then expand Eq. (3.4) to  $\mathcal{O}(2)$  to get

$$(\nabla^2 - m_\psi^2)\psi^{(2)} = 0, \quad (4.11)$$

for the scalar field perturbation  $\psi^{(2)}$ , where  $\nabla^2 = \delta^{ij}\partial_i\partial_j$  and  $m_\psi = 2\kappa\sqrt{\frac{V_2\phi_0}{2\omega_0}}$ . Eq. (4.11) is a screened Poisson equation. Since we demand that  $\phi$  to take its cosmological value at large scale, which is equivalent to saying that the perturbation should vanish at cosmological distance due to the absence of the gravitational field and the matter source, i.e.,  $\psi^{(2)} \rightarrow 0$  as  $r \rightarrow \infty$  ( $r$  is the distance from the Sun), we get the solution of Eq. (4.11) as

$$\psi^{(2)} = 0. \quad (4.12)$$

In order to get the corresponding vierbein perturbations, we use the ansatz

$${}^{(2)}h_{ij} = \gamma(r) {}^{(2)}h_{00}\delta_{ij} = 2\gamma(r)A\delta_{ij}, \quad (4.13)$$

where  $\gamma(r)$  is a PPN parameter measuring the amount of space curvature produced by unit rest mass [73]. We also adopt the gauge conditions for the vierbein perturbation  $B^\mu{}_\nu$  as [91]

$$\partial_j {}^{(2)}B_i{}^j - \frac{1}{2}\partial_i {}^{(2)}B_\mu{}^\mu = \frac{1}{2\phi_0}\partial_i \psi^{(2)} = 0, \quad (4.14a)$$

$$\partial_j {}^{(3)}B_0{}^j - \frac{1}{2}\partial_0 {}^{(2)}B_j{}^j = \frac{1}{2\phi_0}\partial_0 \psi^{(2)} = 0, \quad (4.14b)$$

in which we have used Eq. (4.12) actually. These gauge conditions can directly lead to the standard gauge formulas [92, 93]

$$\partial_j {}^{(2)}h_i{}^j - \frac{1}{2}\partial_i {}^{(2)}h_\mu{}^\mu = \frac{1}{\phi_0}\partial_i \psi^{(2)} = 0, \quad (4.15a)$$

$$\partial_j {}^{(3)}h_0{}^j - \frac{1}{2}\partial_0 {}^{(2)}h_j{}^j = \frac{1}{\phi_0}\partial_0 \psi^{(2)} = 0. \quad (4.15b)$$

We should verify the consistency of these gauge conditions after obtaining the solutions. Actually, as we will see later, our results are identical to GR, so these conditions are just the Newtonian continuity equations [92], and are satisfied automatically.

Expanding (0,0) component of Eq. (3.8) to  $\mathcal{O}(2)$ , we get

$$\phi_0 \left[ \partial_k S_0{}^{k0} - \frac{1}{2}\partial_k S_\rho{}^{k\rho} \right] = -\frac{1}{4}\kappa^2 \rho = \frac{1}{2}\nabla^2 U, \quad (4.16)$$

in which the gravitational potential  $U$  is defined by

$$\nabla^2 U = -\frac{1}{2}\kappa^2 \rho. \quad (4.17)$$

The solution to this equation is

$$A = \frac{U}{\phi_0}. \quad (4.18)$$

Expanding ( $i, j$ ) component of Eq. (3.8) to  $\mathcal{O}(2)$ , we get

$$\phi_0 \left[ \partial_k S_i{}^{kj} - \frac{1}{2}\delta_i^j \partial_k S_\rho{}^{k\rho} \right] = \frac{1}{4}\delta_i^j \kappa^2 \rho. \quad (4.19)$$

Taking the trace of Eq. (4.19) yields

$$\phi_0 \left[ \partial_k S_i{}^{ki} - \frac{3}{2}\partial_k S_\rho{}^{k\rho} \right] = -\frac{3}{2}\nabla^2 U. \quad (4.20)$$

The solution to Eq. (4.20) is given by

$$\gamma(r) = 1. \quad (4.21)$$

Expanding Eq. (3.4) to  $\mathcal{O}(4)$  yields

$$T + 2\frac{\omega_0}{\phi_0}\nabla^2 \psi^{(4)} - 4\kappa^2 V_2 \psi^{(4)} = 0. \quad (4.22)$$

Noting that  $T = 2\partial_i A \partial^i A$  (see Eq. (A28)), the above equation can be simplified to

$$(\nabla^2 - m_\psi^2) \psi^{(4)} = \frac{1}{\phi_0 \omega_0} \nabla^2 \left( \Phi_2 - \frac{U^2}{2} \right), \quad (4.23)$$

where we have used the identity

$$\partial_i U \partial^i U = \frac{1}{2} \nabla^2 U^2 - U \nabla^2 U, \quad (4.24)$$

and  $\Phi_2$  is defined by

$$\nabla^2 \Phi_2 = -\frac{\kappa^2}{2} \rho U. \quad (4.25)$$

Eq. (4.23) is a screened Poisson equation and can be solved by

$$\psi^{(4)} = \frac{1}{\phi_0 \omega_0} \left( \Phi_2 - \frac{U^2}{2} \right) e^{-m_\psi r}. \quad (4.26)$$

Expanding  $(0, i)$  component of Eq. (3.8) to  $\mathcal{O}(3)$ , we obtain

$$\phi_0 e^{-1} e^a{}_0 \partial_\sigma (e S_a{}^{\sigma i}) = \frac{\kappa^2}{2} (-\rho v^i). \quad (4.27)$$

The solution to this equation is

$${}^{(3)}B_{0i} = -\frac{1}{\phi_0} \left( \frac{7}{4} \mathcal{V}_i + \frac{1}{4} W_i \right), \quad (4.28)$$

with  $\mathcal{V}_i$  and  $W_i$  defined as in [90],

$$\nabla^2 \mathcal{V}_i = -\frac{\kappa^2}{2} \rho v_i, \quad (4.29)$$

and

$$W_i = G_N \int d^3 y \frac{\rho(y, t) v^k(y, t) (x - y)_k (x - y)_i}{|x - y|^3}. \quad (4.30)$$

Note that we have used the fact  $2\partial_0 \partial_i (\phi_0 A) = -\nabla^2 (\mathcal{V}_i - W_i)$  [90] to derive Eq. (4.28).

Expanding  $(0, 0)$  component of Eq. (3.8) to  $\mathcal{O}(4)$ , we obtain

$$\phi_0 e^{-1} e^a{}_0 \partial_\sigma (e S_a{}^{\sigma 0}) - \frac{1}{2} \phi_0 e^{-1} e^a{}_\rho \partial_\sigma (e S_a{}^{\sigma \rho}) + \phi_0 \left( T^\sigma{}_{\alpha 0} S^\sigma{}^{\alpha 0} + \frac{T}{4} \right) = \frac{\kappa^2}{2} \left( \mathcal{T}_0{}^0 - \frac{1}{2} \mathcal{T} \right). \quad (4.31)$$

The solution to this equation is

$$B_{00} = \frac{1}{\phi_0} (U + 2\Phi_1 + \Phi_3 + 3\Phi_4) + \frac{2}{\phi_0^2} \Phi_2 - \frac{1}{2\phi_0^2} U^2, \quad (4.32)$$

where  $\Phi_1$ ,  $\Phi_3$ , and  $\Phi_4$  are defined as in [73],

$$\nabla^2 \Phi_1 = -\frac{\kappa^2}{2} \rho v^2, \quad (4.33a)$$

$$\nabla^2 \Phi_3 = -\frac{\kappa^2}{2} \rho \Pi, \quad (4.33b)$$

$$\nabla^2 \Phi_4 = -\frac{\kappa^2}{2} p. \quad (4.33c)$$

In summary, we get the corresponding metric perturbations as

$$h_{00} = \frac{2}{\phi_0} (U + 2\Phi_1 + \Phi_3 + 3\Phi_4) + \frac{4}{\phi_0^2} \Phi_2 - \frac{2}{\phi_0^2} U^2, \quad (4.34a)$$

$$h_{0j} = -\frac{2}{\phi_0} \left( \frac{7}{4} \mathcal{V}_j + \frac{1}{4} W_j \right), \quad (4.34b)$$

$$h_{ij} = 2 \frac{U}{\phi_0} \delta_{ij}. \quad (4.34c)$$



From above equations, it is easy to see that the effective Newtonian constant  $G_{\text{eff}} = G_N/\phi_0$ , and the PPN parameter  $\beta(r)$  is given by

$$\beta(r) = 1. \quad (4.35)$$

We note that the PPN parameter  $\beta(r)$  measures the amount of “non-linearity” in the superposition law for gravity [73]. Notice that Eqs. (4.21) and (4.35) are the main results of this work.

## V. CONCLUSIONS AND DISCUSSIONS

We have studied the post-Newtonian approximation of teleparallel gravity coupling to a scalar field  $\phi$  with arbitrary coupling function  $\omega(\phi)$  and arbitrary potential  $V(\phi)$ . We have chosen frames in which the Sun is at rest in both the coordinate frame and the tetrad frame, such that the vierbein (dual vierbein) can be perturbatively expanded around the flat spacetime, which leads to the usual expanding of the metric around the Minkowski spacetime. The functions  $\omega(\phi)$  and  $V(\phi)$  are characterized by the coefficients of Taylor expansion. Interestingly, the only non-vanishing PPN parameters  $\beta$  and  $\gamma$  are all equal to 1, indicating that these models are indistinguishable from GR in the Solar System distance up to the post-Newtonian order. In addition, we can rescale the cosmological background value  $\phi_0$  of the scalar to  $\phi_0 = 1$ , and then  $G_{\text{eff}} = G_N$ . Since the rescaling can be done globally, we conclude that the effective Newtonian constant has no contribution to the Solar System experiments neither.

This feature makes the theories we studied quite different from the scalar-tensor theories (nb. [100]), which might be subject to stringent constraints on the parameter space, or need some screening mechanisms to pass the Solar System experimental constraints. We might conclude that the coupling between the scalar field and the torsion scalar in teleparallel gravity is less strong as that between the scalar and the Ricci scalar in GR. This can be seen from the relationship between the torsion scalar constructed from the Weitzenböck connection and the Ricci scalar constructed from the Levi-Civita connection [94],

$$T = -R - 2\nabla^\mu T^\nu_{\mu\nu}. \quad (5.1)$$

Although the second term on the right hand side of Eq. (5.1) is a boundary term in the TEGR case, it will be nontrivial when a scalar field  $\phi$  is coupled to the torsion

$$\phi T = -\phi R - 2\phi\nabla^\mu T^\nu_{\mu\nu}, \quad (5.2)$$

which makes the theories quite different from the scalar-tensor theories. In addition,  $T$  is at least  $\mathcal{O}(4)$ , while  $R$  is at least  $\mathcal{O}(2)$  when perturbed around the flat spacetime. This fact makes the gravitational sector have no effect on the  $\psi^{(2)}$  when Eq. (3.4) is expanded up to  $\mathcal{O}(2)$ , thus leading to the PPN parameter  $\gamma(r)$  equals to 1. This result is agree with the previous work in [67]. The authors in [67] have argued that, since the source matter is not involved in the solution of  $\mathcal{O}(2)$  perturbation of the scalar field (see Eq. (3.7) in our case), the Newtonian potential cannot be modified to a Yukawa type  $U(r) = Ue^{-mr}$  as in scalar-tensor theories. Although the non-minimally coupling between the scalar and the torsion shows no deviation from GR in the post-Newtonian approximation, the distinction may appear in the post-post-Newtonian [101] limit when such experiments are available. In fact, the scalar perturbation  $\psi^{(4)}$  is non-vanishing (see Eq. (4.26)), and it definitely will affect the post-post-Newtonian behavior through Eq. (3.8). This indirect coupling between the scalar field and the gravitational sector is the meaning of less strong coupling we proposed.

Similar to  $f(T)$  theory [102], the action (3.1) is not invariant under local Lorentz transformation. One might, therefore, expect some preferred-frame effects to show up in post-Newtonian limit (we thank the referee for pointing out this issue). Although our results reveal no coordinate frame is preferred in obtaining PPN parameters, there is indeed a preferred tetrad frame in our calculation. It is interesting to note that similar results have been achieved in some scalar-tetrad theories of gravity (see e.g. [103, 104] and the references therein). It is claimed in [103, 104] that the preferred-tetrad-frame effect cannot be detected if one only measures the metric components. Attempts to measure the tetrad in a direct way, e.g., the interaction of tetrad with a spin-1/2 field, would generally introduce some Lorentz gauge fields to restore the local Lorentz symmetry [47], and thus creating a Poincaré gauge theory. We refer to [103, 104] for a more detailed discussion of this issue.

We stress here that not all kinds of non-minimally coupling between the torsion and the scalar would have no affect on the weak field behavior of the theory (we are indebted to an anonymous colleague for pointing out this issue). For example, if we add a term of the form  $T^\alpha_{\alpha\beta} \partial^\beta \phi$  as considered in e.g. [95, 96] to the action (3.1), an extra term like

$$- \partial^k T^\alpha_{\alpha\beta} \quad (5.3)$$

would be added to the  $\mathcal{O}(2)$  perturbative equation of  $\phi$  (i.e. Eq. (4.11)). Therefore, the value of  $\psi^{(2)}$  will not vanish in this case, thus changing the gauge conditions (4.14). So, quite contrary to our original action (3.1), the additional non-minimally coupling term  $T^\alpha_{\alpha\beta} \partial^\beta \phi$  might make the PPN parameters differ from the case of GR. We leave this issue to the future works.

One might note that the action (3.1) considered in this work could be further generalized to

$$S = \frac{1}{2\kappa^2} \int d^4x e \left[ \xi(\phi) T - \frac{\omega(\phi)}{\phi} (\partial\phi)^2 - 2\kappa^2 V(\phi) \right] + S_m [e_a^\mu, \chi_m]. \quad (5.4)$$

However, it is an illusion. Introducing a new scalar  $\hat{\phi} = \xi(\phi)$ , Eq. (5.4) can be recast as

$$\hat{S} = \frac{1}{2\kappa^2} \int d^4x e \left[ \hat{\phi} T - \frac{\hat{\omega}(\hat{\phi})}{\hat{\phi}} (\partial\hat{\phi})^2 - 2\kappa^2 \hat{V}(\hat{\phi}) \right] + S_m [e_a^\mu, \chi_m], \quad (5.5)$$

which reduces to the action (3.1) actually. So, the conclusions do not change for the action (5.4). This indicates that the action (3.1) considered in this work is general enough.

Although the theories we studied here have the same PPN parameters as GR, it differs from GR in several aspects. Firstly, the deviation from GR might show up in the higher order perturbation, e.g. in the post-post-Newtonian limit [101]. Secondly, we should consider the preferred tetrad frame effect (we thank the referee for pointing out this issue). Unfortunately, there are no PPN parameters to characterize this effect. So, we get the same PPN parameters as GR. The standard post-Newtonian formalism might be generalized to incorporate this effect. And it is beyond the scope of the present work.

Finally, from the viewpoint of symmetry, black holes have similar environments like the Solar System. So, we might speculate that our theories will have the same solutions as GR when applying to black holes. Thus, it would be interesting to study the black hole solutions in the future works.

### Acknowledgments

We thank the anonymous referee for quite useful comments and suggestions, which helped us to improve this work. We are indebted to an anonymous colleague for pointing out the references [95, 96]. We are grateful to Jing Liu, Xiao-Peng Yan, Ya-Nan Zhou, Xiao-Bo Zou, and Hong-Yu Li for kind help and discussions. This work was supported in part by NSFC under Grants No. 11175016 and No. 10905005, as well as NCET under Grant No. NCET-11-0790.

### Appendix A: Perturbations for the torsion tensor and the super-potential tensor

We present here the detailed calculations of the perturbations for the torsion tensor  $T^\lambda_{\mu\nu}$  and the super-potential tensor  $S^\rho_{\mu\nu}$  which are constructed from the vierbein  $e_a^\mu$  and dual vierbein  $e^a_\mu$  to sufficient order. Note that the ansatz (4.13) is equivalent to

$$^{(2)}B_{ij} = \gamma A \delta_{ij}, \quad ^{(2)}B_{00} = A. \quad (A1)$$

After solving the  $(i, j)$  component of Eq. (3.4) to  $\mathcal{O}(2)$ , we have the PPN parameter  $\gamma(r)$  equal to 1 (nb. Eq. (4.21)). So, when dealing with  $\mathcal{O}(2)$  quantities, we explicitly show the parameter  $\gamma$  in the expansion, and simplify  $\mathcal{O}(3)$  and  $\mathcal{O}(4)$  quantities by setting  $\gamma = 1$ .

We firstly expand the vierbein fields around the flat background as

$$e_a^\mu = \delta_a^\mu + C_a^\mu = \delta_a^\mu + {}^{(2)}C_a^\mu + {}^{(3)}C_a^\mu + {}^{(4)}C_a^\mu + \mathcal{O}(5), \quad (\text{A2})$$

where each term  ${}^{(n)}C_a^\mu$  is of order  $\mathcal{O}(n)$ . Noting that  $g^{\mu\nu}(x) = \eta^{ab}e_a^\mu(x)e_b^\nu(x)$  and using Eq. (2.1), we can easily get  ${}^{(2)}C_a^\lambda = -{}^{(2)}B_a^\lambda$ . We then expand the torsion tensor  $T_{\mu\nu}^\lambda$  up to  $\mathcal{O}(4)$ ,

$$\begin{aligned} T_{\mu\nu}^\lambda &= e_a^\lambda (\partial_\nu e_\mu^a - \partial_\mu e_\nu^a) \\ &= (\delta_a^\lambda + C_a^\lambda) (\partial_\nu B_\mu^a - \partial_\mu B_\nu^a) \\ &= \partial_\nu B_\mu^\lambda - \partial_\mu B_\nu^\lambda + C_a^\lambda \partial_\nu B_\mu^a - C_a^\lambda \partial_\mu B_\nu^a \\ &= \partial_\nu B_\mu^\lambda - \partial_\mu B_\nu^\lambda - {}^{(2)}B_\alpha^\lambda \partial_\nu {}^{(2)}B_\mu^\alpha + {}^{(2)}B_\alpha^\lambda \partial_\mu {}^{(2)}B_\nu^\alpha. \end{aligned} \quad (\text{A3})$$

For convenience, we also present the definition of the super-potential tensor  $S_{\mu\nu}^\rho$  here,

$$S_{\mu\nu}^\rho = \frac{1}{4}(T_{\mu\nu}^\rho - T_{\mu\nu}^\rho + T_{\nu\mu}^\rho) + \frac{1}{2}\delta_\mu^\rho T_{\nu\sigma}^\sigma - \frac{1}{2}\delta_\nu^\rho T_{\mu\sigma}^\sigma. \quad (\text{A4})$$

In addition, we use the anti-symmetric properties of the torsion tensor  $T_{\mu\nu}^\lambda$  and the super-potential tensor  $S_{\mu\nu}^\rho$  to simplify our calculations. Since the space-space component of metric  $g_{ij}$  is expanded around the usual Euclidean metric  $\delta_{ij}$ , we do not distinguish the upper indices and the lower indices of the perturbation quantities up to appropriate order. Instead, we use the upper indices and the lower indices interchangeably, e.g.  ${}^{(2)}B_j^i = {}^{(2)}B_{ij} = {}^{(2)}B_i^j = {}^{(2)}B^{ij}$ , up to  $\mathcal{O}(2)$ .

### 1. Up to $\mathcal{O}(2)$

The expansion of torsion tensor to  $\mathcal{O}(2)$  can be read from Eq. (A3) as

$$T_{\mu\nu}^\lambda = \partial_\nu {}^{(2)}B_\mu^\lambda - \partial_\mu {}^{(2)}B_\nu^\lambda. \quad (\text{A5})$$

Some of its components can be obtained directly,

$$T_{i0}^0 = \partial_0 {}^{(2)}B_i^0 - \partial_i {}^{(2)}B_0^0 = \partial_i A, \quad (\text{A6})$$

$$T_{00}^k = \partial_0 {}^{(2)}B_0^k - \partial_0 {}^{(2)}B_0^k = 0, \quad (\text{A7})$$

$$T_{ij}^0 = \partial_j {}^{(2)}B_i^0 - \partial_i {}^{(2)}B_j^0 = 0, \quad (\text{A8})$$

$$T_{ji}^k = \partial_i {}^{(2)}B_j^k - \partial_j {}^{(2)}B_i^k = \delta_j^k \partial_i (\gamma A) - \delta_i^k \partial_j (\gamma A), \quad (\text{A9})$$

$$T_{ij}^j = \partial_j {}^{(2)}B_i^j - \partial_i {}^{(2)}B_j^j = -2\partial_i (\gamma A), \quad (\text{A10})$$

$$T_{0j}^i = \partial_j {}^{(2)}B_0^i - \partial_0 {}^{(2)}B_j^i = 0. \quad (\text{A11})$$

And the expansion for some components of the super-potential tensor  $S_{\mu\nu}^\rho$  is also obtained,

$$\begin{aligned} S_{i0}^0 &= \frac{1}{4}(T_{i0}^0 - T_{i0}^0 + T_{0i}^0) - \frac{1}{2}T_{i\sigma}^\sigma \\ &= \frac{1}{4}(T_{i0}^0 + T_{00}^i + T_{i0}^0) - \frac{1}{2}(T_{i0}^0 + T_{ij}^j) = -\frac{1}{2}T_{ij}^j = \partial_i (\gamma A), \end{aligned} \quad (\text{A12})$$

$$\begin{aligned} S_{ij}^j &= \frac{1}{4}(T_{ij}^j - T_{ij}^j + T_{ji}^j) + \frac{1}{2}\delta_i^j T_{j\sigma}^\sigma - \frac{1}{2}\delta_j^j T_{i\sigma}^\sigma = \frac{1}{4}(T_{ij}^j - T_{jj}^i + T_{ij}^j) + \frac{1}{2}T_{i\sigma}^\sigma - \frac{3}{2}T_{i\sigma}^\sigma \\ &= \frac{1}{2}T_{ij}^j - (T_{i0}^0 + T_{ij}^j) = -\frac{1}{2}T_{ij}^j - T_{i0}^0 = \partial_i (\gamma A) - \partial_i A, \end{aligned} \quad (\text{A13})$$

$$S^i_{0j} = \frac{1}{4}(T^i_{0j} - T_{0j}{}^i + T_{j0}{}^i) - \frac{1}{2}\delta_j^i T^\sigma_{0\sigma} = \frac{1}{4}(T^i_{0j} + T^0_{ji} + T^j_{0i}) - \frac{1}{2}\delta_j^i (T^0_{00} + T^k_{0k}) = 0. \quad (\text{A14})$$

We also present the result of  $\partial_\mu e$  up to  $\mathcal{O}(2)$  here,

$$\partial_\mu e = \partial_\mu \sqrt{-g} = \partial_\mu \left( 1 + \frac{1}{2} {}^{(2)}h^\nu{}_\nu \right) = \frac{1}{2} \partial_\mu {}^{(2)}h^\nu{}_\nu = \partial_\mu (3\gamma A - A), \quad (\text{A15})$$

where we have used the fact that

$${}^{(2)}h^\nu{}_\nu = {}^{(2)}h^0{}_0 + {}^{(2)}h^i{}_i = -{}^{(2)}h_{00} + {}^{(2)}h_{ii} = -2 {}^{(2)}B_{00} + 2 {}^{(2)}B_{ii} = -2A + 6\gamma A. \quad (\text{A16})$$

## 2. Up to $\mathcal{O}(3)$

The expansion of torsion tensor to  $\mathcal{O}(3)$  can be read from Eq. (A3) as

$$T^\lambda_{\mu\nu} = \partial_\nu B^\lambda{}_\mu - \partial_\mu B^\lambda{}_\nu. \quad (\text{A17})$$

Some of its components read

$$T^0_{ij} = \partial_j B^0{}_i - \partial_i B^0{}_j = \partial_j {}^{(3)}B^0{}_i - \partial_i {}^{(3)}B^0{}_j, \quad (\text{A18})$$

$$T^i_{j0} = \partial_0 B^i{}_j - \partial_j B^i{}_0 = \partial_0 {}^{(2)}B^i{}_j - \partial_j {}^{(3)}B^i{}_0 = \delta_j^i \partial_0 A - \partial_j {}^{(3)}B^i{}_0, \quad (\text{A19})$$

$$T^i_{i0} = 3\partial_0 A - \partial_i {}^{(3)}B^i{}_0 = \frac{3}{2} \partial_0 A. \quad (\text{A20})$$

When we derive above equations, the gauge conditions (4.14) have been used. Some components of the super-potential read

$$S^0_{ij} = \frac{1}{4}(T^0_{ij} - T_{ij}{}^0 + T_{ji}{}^0) = \frac{1}{4}(T^0_{ij} + T^i_{j0} - T^j_{i0}) = \frac{1}{2}(\partial_j {}^{(3)}B^0{}_i - \partial_i {}^{(3)}B^0{}_j), \quad (\text{A21})$$

$$\begin{aligned} S^i_{0i} &= \frac{1}{4}(T^i_{0i} - T_{0i}{}^i + T_{i0}{}^i) - \frac{1}{2}\delta_i^i T^\sigma_{0\sigma} \\ &= \frac{1}{4}(T^i_{0i} + T^0_{ii} + T^i_{0i}) - \frac{3}{2}(T^0_{00} + T^i_{0i}) = T^i_{i0} = \frac{3}{2} \partial_0 A. \end{aligned} \quad (\text{A22})$$

## 3. Up to $\mathcal{O}(4)$

The expansion of torsion tensor to  $\mathcal{O}(4)$  can be read from Eq. (A3) as

$$T^\lambda_{\mu\nu} = \partial_\nu B^\lambda{}_\mu - \partial_\mu B^\lambda{}_\nu - {}^{(2)}B^\lambda{}_\alpha \partial_\nu {}^{(2)}B^\alpha{}_\mu + {}^{(2)}B^\lambda{}_\alpha \partial_\mu {}^{(2)}B^\alpha{}_\nu, \quad (\text{A23})$$

which can directly lead to

$$\begin{aligned} T^0_{i0} &= \partial_0 B^0{}_i - \partial_i B^0{}_0 - {}^{(2)}B^0{}_\alpha \partial_0 {}^{(2)}B^\alpha{}_i + {}^{(2)}B^0{}_\alpha \partial_i {}^{(2)}B^\alpha{}_0 \\ &= \partial_0 {}^{(3)}B^0{}_i - \partial_i B^0{}_0 + {}^{(2)}B^0{}_0 \partial_i {}^{(2)}B^0{}_0 = \partial_0 {}^{(3)}B^0{}_i - \partial_i B^0{}_0 + A \partial_i A, \end{aligned} \quad (\text{A24})$$

$$\begin{aligned} T^j_{ij} &= \partial_j B^j{}_i - \partial_i B^j{}_j - {}^{(2)}B^j{}_\alpha \partial_j {}^{(2)}B^\alpha{}_i + {}^{(2)}B^j{}_\alpha \partial_i {}^{(2)}B^\alpha{}_j \\ &= \partial_j {}^{(2)}B^j{}_i - \partial_i {}^{(2)}B^j{}_j - {}^{(2)}B^j{}_k \partial_j {}^{(2)}B^k{}_i + {}^{(2)}B^j{}_k \partial_i {}^{(2)}B^k{}_j = -2\partial_i A + 2A \partial_i A. \end{aligned} \quad (\text{A25})$$

The components of the super-potential for our interest are also given,

$$\begin{aligned} S^0_{i0} &= \frac{1}{4}(T^0_{i0} - T_{i0}{}^0 + T_{0i}{}^0) - \frac{1}{2}T^\sigma_{i\sigma} \\ &= \frac{1}{4}(T^0_{i0} + T^0_{i0}) - \frac{1}{2}(T^0_{i0} + T^j_{ij}) = -\frac{1}{2}T^j_{ij} = \partial_i A - A \partial_i A, \end{aligned} \quad (\text{A26})$$

$$\begin{aligned}
S^i_{ji} &= \frac{1}{4}(T^i_{ji} - T_{ji}^i + T_{ij}^i) + \frac{1}{2}\delta_j^i T^\sigma_{i\sigma} - \frac{1}{2}\delta_i^j T^\sigma_{j\sigma} = \frac{1}{4}(T^i_{ji} + T^i_{ji}) + \frac{1}{2}T^\sigma_{j\sigma} - \frac{3}{2}T^\sigma_{j\sigma} \\
&= \frac{1}{2}T^i_{ji} - (T^0_{j0} + T^i_{ji}) = -\frac{1}{2}T^i_{ji} - T^0_{j0} = -\partial_0^{(3)}B^0_j + \partial_j B^0_0 - 2A\partial_j A + \partial_j A.
\end{aligned} \tag{A27}$$

Finally, we expand the torsion scalar  $T$  up to  $\mathcal{O}(4)$  as

$$\begin{aligned}
T &= S^\rho_{\mu\nu} T^\rho_{\mu\nu} = S^\rho_{\mu 0} T^\rho_{\mu 0} + S^\rho_{\mu i} T^\rho_{\mu i} = S^\rho_{00} T^\rho_{00} + S^\rho_{i0} T^\rho_{i0} + S^\rho_{0i} T^\rho_{0i} + S^\rho_{ji} T^\rho_{ji} \\
&= S^0_{i0} T^0_{i0} + S^j_{i0} T^j_{i0} + S^0_{0i} T^0_{0i} + S^j_{0i} T^j_{0i} + S^0_{ji} T^0_{ji} + S^k_{ji} T^k_{ji} \\
&= S^0_{i0} T^0_{i0} + S^0_{0i} T^0_{0i} + S^k_{ji} T^k_{ji} = 2S^0_{i0} T^0_{i0} + S^k_{ji} (\delta_j^k \partial_i A - \delta_i^k \partial_j A) \\
&= 2S^0_{i0} T^0_{i0} = 2\partial_i A \partial^i A.
\end{aligned} \tag{A28}$$

- 
- [1] A. G. Riess *et al.* (Supernova Search Team), *Astron. J.* **116**, 1009 (1998), [astro-ph/9805201](#).
  - [2] S. Perlmutter *et al.* (Supernova Cosmology Project), *Astrophys. J.* **517**, 565 (1999), [astro-ph/9812133](#).
  - [3] D. Spergel *et al.* (WMAP Collaboration), *Astrophys. J. Suppl.* **148**, 175 (2003), [astro-ph/0302209](#).
  - [4] M. Tegmark *et al.* (SDSS Collaboration), *Phys. Rev.* **D69**, 103501 (2004), [astro-ph/0310723](#).
  - [5] S. Allen *et al.*, *Mon. Not. Roy. Astron. Soc.* **353**, 457 (2004), [astro-ph/0405340](#).
  - [6] A. G. Riess *et al.* (Supernova Search Team), *Astrophys. J.* **607**, 665 (2004), [astro-ph/0402512](#).
  - [7] D. Spergel *et al.* (WMAP Collaboration), *Astrophys. J. Suppl.* **170**, 377 (2007), [astro-ph/0603449](#).
  - [8] P. Ade *et al.* (Planck Collaboration), *Astron. Astrophys.* **571**, A1 (2014), [arXiv:1303.5062 \[astro-ph.CO\]](#).
  - [9] P. Ade *et al.* (Planck Collaboration), *Astron. Astrophys.* **571**, A16 (2014), [arXiv:1303.5076 \[astro-ph.CO\]](#).
  - [10] S. Weinberg, *Rev. Mod. Phys.* **61**, 1 (1989).
  - [11] S. M. Carroll, W. H. Press, and E. L. Turner, *Ann. Rev. Astron. Astrophys.* **30**, 499 (1992).
  - [12] I. Zlatev, L.-M. Wang, and P. J. Steinhardt, *Phys. Rev. Lett.* **82**, 896 (1999), [astro-ph/9807002](#).
  - [13] S. M. Carroll, *Living Rev. Rel.* **4**, 1 (2001), [astro-ph/0004075](#).
  - [14] P. Peebles and B. Ratra, *Rev. Mod. Phys.* **75**, 559 (2003), [astro-ph/0207347](#).
  - [15] T. Padmanabhan, *Phys. Rept.* **380**, 235 (2003), [hep-th/0212290](#).
  - [16] J. Sola, *J. Phys. Conf. Ser.* **453**, 012015 (2013), [arXiv:1306.1527 \[gr-qc\]](#).
  - [17] V. L. Fitch *et al.*, eds., *Critical Problems in Physics* (Princeton University Press, 1997).
  - [18] E. J. Copeland, M. Sami, and S. Tsujikawa, *Int. J. Mod. Phys.* **D15**, 1753 (2006), [hep-th/0603057](#).
  - [19] D. H. Weinberg *et al.*, *Phys. Rept.* **530**, 87 (2013), [arXiv:1201.2434 \[astro-ph.CO\]](#).
  - [20] Y. Wang, *Dark Energy* (Wiley-VCH, 2010).
  - [21] R. Caldwell, R. Dave, and P. J. Steinhardt, *Phys. Rev. Lett.* **80**, 1582 (1998), [astro-ph/9708069](#).
  - [22] L. Ford, *Phys. Rev.* **D35**, 2339 (1987).
  - [23] B. Ratra and P. Peebles, *Phys. Rev.* **D37**, 3406 (1988).
  - [24] R. Caldwell, *Phys. Lett.* **B545**, 23 (2002), [astro-ph/9908168](#).
  - [25] R. R. Caldwell *et al.*, *Phys. Rev. Lett.* **91**, 071301 (2003), [astro-ph/0302506](#).
  - [26] C. Armendariz-Picon *et al.*, *Phys. Rev.* **D63**, 103510 (2001), [astro-ph/0006373](#).
  - [27] T. Chiba, T. Okabe, and M. Yamaguchi, *Phys. Rev.* **D62**, 023511 (2000), [astro-ph/9912463](#).
  - [28] C. Armendariz-Picon *et al.*, *Phys. Lett.* **B458**, 209 (1999), [hep-th/9904075](#).
  - [29] J. Garriga and V. F. Mukhanov, *Phys. Lett.* **B458**, 219 (1999), [hep-th/9904176](#).
  - [30] E. J. Copeland *et al.*, *Phys. Rev.* **D71**, 043003 (2005), [hep-th/0411192](#).
  - [31] J. Bagla, H. K. Jassal, and T. Padmanabhan, *Phys. Rev.* **D67**, 063504 (2003), [astro-ph/0212198](#).
  - [32] G. W. Gibbons, *Phys. Lett.* **B537**, 1 (2002), [hep-th/0204008](#).
  - [33] L. R. W. Abramo and F. Finelli, *Phys. Lett.* **B575**, 165 (2003), [astro-ph/0307208](#).
  - [34] T. Padmanabhan, *Phys. Rev.* **D66**, 021301 (2002), [hep-th/0204150](#).
  - [35] T. Vachaspati and M. Trodden, *Phys. Rev.* **D61**, 023502 (1999), [gr-qc/9811037](#).
  - [36] T. Clifton *et al.*, *Phys. Rept.* **513**, 1 (2012), [arXiv:1106.2476 \[astro-ph.CO\]](#).
  - [37] S. Tsujikawa, “Modified Gravity Models of Dark Energy,” in *Lectures on Cosmology*, Lecture Notes in Physics No. 800 (Springer Berlin Heidelberg, 2010) pp. 99–145.

- [38] P. G. Bergmann, *Int. J. Theor. Phys.* **1**, 25 (1968).
- [39] R. V. Wagoner, *Phys. Rev.* **D1**, 3209 (1970).
- [40] G. Esposito-Farese and D. Polarski, *Phys. Rev.* **D63**, 063504 (2001), [gr-qc/0009034](#) .
- [41] V. Faraoni, *Cosmology in Scalar-Tensor Gravity* (Springer, 2004).
- [42] Y. Fujii and K.-i. Maeda, *The Scalar-Tensor Theory of Gravitation* (Cambridge University Press, 2004).
- [43] N. Bartolo and M. Pietroni, *Phys. Rev.* **D61**, 023518 (2000), [hep-ph/9908521](#) .
- [44] B. Boisseau, *Phys. Rev.* **D83**, 043521 (2011), [arXiv:1011.2915 \[astro-ph.CO\]](#) .
- [45] C. Charmousis *et al.*, *Phys. Rev. Lett.* **108**, 051101 (2012), [arXiv:1106.2000 \[hep-th\]](#) .
- [46] F. W. Hehl, P. von der Heyde, G. D. Kerlick, and J. M. Nester, *Rev. Mod. Phys.* **48**, 393 (1976).
- [47] F. W. Hehl, J. Nitsch, and P. von der Heyde, in *General Relativity and Gravitation. Vol. 1. One hundred years after the birth of Albert Einstein* (New York, NY: Plenum Press, 1980).
- [48] R. Weitzenböck, *Invariantentheorie* (Groningen: P. Noordhoff, 1923).
- [49] A. Einstein, Sitzber. Preuss. Akad. Wiss. **17**, 217 (1928).
- [50] A. Einstein, Sitzber. Preuss. Akad. Wiss. **17**, 224 (1928).
- [51] A. Einstein, *Math. Annal.* **102**, 685 (1930).
- [52] A. Unzicker and T. Case, [physics/0503046](#) .
- [53] K. Hayashi and T. Nakano, *Prog. Theor. Phys.* **38**, 491 (1967).
- [54] Y. M. Cho, *Phys. Rev.* **D14**, 2521 (1976).
- [55] Y. M. Cho, *Phys. Rev.* **D14**, 3335 (1976).
- [56] K. Hayashi, *Phys. Lett.* **B69**, 441 (1977).
- [57] K. Hayashi and T. Shirafuji, *Phys. Rev.* **D19**, 3524 (1979).
- [58] G. R. Bengochea and R. Ferraro, *Phys. Rev.* **D79**, 124019 (2009), [arXiv:0812.1205 \[astro-ph\]](#) .
- [59] E. V. Linder, *Phys. Rev.* **D81**, 127301 (2010), [arXiv:1005.3039 \[astro-ph.CO\]](#) .
- [60] C.-Q. Geng *et al.*, *Phys. Lett.* **B704**, 384 (2011), [arXiv:1109.1092 \[hep-th\]](#) .
- [61] C.-Q. Geng, C.-C. Lee, and E. N. Saridakis, *JCAP* **1201**, 002 (2012), [arXiv:1110.0913 \[astro-ph.CO\]](#) .
- [62] H. Wei, *Phys. Lett.* **B712**, 430 (2012), [arXiv:1109.6107 \[gr-qc\]](#) .
- [63] C. Xu, E. N. Saridakis, and G. Leon, *JCAP* **1207**, 005 (2012), [arXiv:1202.3781 \[gr-qc\]](#) .
- [64] C.-Q. Geng and Y.-P. Wu, *JCAP* **1304**, 033 (2013), [arXiv:1212.6214 \[astro-ph.CO\]](#) .
- [65] C.-Q. Geng, J.-A. Gu, and C.-C. Lee, *Phys. Rev.* **D88**, 024030 (2013), [arXiv:1306.0333 \[astro-ph.CO\]](#) .
- [66] J.-A. Gu, C.-C. Lee, and C.-Q. Geng, *Phys. Lett.* **B718**, 722 (2013), [arXiv:1204.4048 \[astro-ph.CO\]](#) .
- [67] J.-T. Li, Y.-P. Wu, and C.-Q. Geng, *Phys. Rev.* **D89**, 044040 (2014), [arXiv:1312.4332 \[gr-qc\]](#) .
- [68] A. Banijamali and B. Fazlpour, *Astrophys. Space Sci.* **342**, 229 (2012), [arXiv:1206.3580 \[physics.gen-ph\]](#) .
- [69] G. Otalora, *Phys. Rev.* **D88**, 063505 (2013), [arXiv:1305.5896 \[gr-qc\]](#) .
- [70] B. Fazlpour and A. Banijamali, [arXiv:1408.0203 \[gr-qc\]](#) .
- [71] A. Banijamali, *Adv. High Energy Phys.* **2014**, 631630 (2014).
- [72] Y. Kucukakca, *Eur. Phys. J.* **C73**, 2327 (2013), [arXiv:1404.7315 \[gr-qc\]](#) .
- [73] C. M. Will, *Theory and Experiment in Gravitational Physics*, Revised ed. (Cambridge Univ. Press, 1993).
- [74] C. Brans and R. H. Dicke, *Phys. Rev.* **124**, 925 (1961).
- [75] Y.-C. Li, F.-Q. Wu, and X. Chen, *Phys. Rev.* **D88**, 084053 (2013), [arXiv:1305.0055 \[astro-ph.CO\]](#) .
- [76] B. Bertotti, L. Iess, and P. Tortora, *Nature* **425**, 374 (2003).
- [77] M. Soussa and R. Woodard, *Gen. Rel. Grav.* **36**, 855 (2004), [astro-ph/0308114](#) .
- [78] T. Chiba, *Phys. Lett.* **B575**, 1 (2003), [astro-ph/0307338](#) .
- [79] G. J. Olmo, *Phys. Rev. Lett.* **95**, 261102 (2005), [gr-qc/0505101](#) .
- [80] J. Khoury and A. Weltman, *Phys. Rev.* **D69**, 044026 (2004), [astro-ph/0309411](#) .
- [81] J. Khoury and A. Weltman, *Phys. Rev. Lett.* **93**, 171104 (2004), [astro-ph/0309300](#) .
- [82] S. Tsujikawa *et al.*, *Phys. Rev.* **D77**, 103009 (2008), [arXiv:0803.1106 \[astro-ph\]](#) .
- [83] R. Kase and S. Tsujikawa, *JCAP* **1308**, 054 (2013), [arXiv:1306.6401 \[gr-qc\]](#) .
- [84] A. Vainshtein, *Phys. Lett.* **B39**, 393 (1972).
- [85] C. Deffayet *et al.*, *Phys. Rev.* **D65**, 044026 (2002), [hep-th/0106001](#) .
- [86] N. Arkani-Hamed, H. Georgi, and M. D. Schwartz, *Annals Phys.* **305**, 96 (2003), [hep-th/0210184](#) .
- [87] K. Hinterbichler and J. Khoury, *Phys. Rev. Lett.* **104**, 231301 (2010), [arXiv:1001.4525 \[hep-th\]](#) .
- [88] K. Hinterbichler *et al.*, *Phys. Rev.* **D84**, 103521 (2011), [arXiv:1107.2112 \[astro-ph.CO\]](#) .
- [89] G. Otalora, *JCAP* **1307**, 044 (2013), [arXiv:1305.0474 \[gr-qc\]](#) .
- [90] S. Chandrasekhar, *Astrophys. J.* **142**, 1488 (1965).
- [91] L. L. Smalley, *Phys. Rev.* **D21**, 328 (1980).

- [92] Y. Nutku, *Astrophys. J.* **155**, 999 (1969).
- [93] C. M. Will, *Astrophys. J.* **163**, 611 (1971).
- [94] J. Maluf, *J. Math. Phys.* **35**, 335 (1994).
- [95] D. Sáez, *Phys. Rev.* **D27**, 2839 (1983).
- [96] D. Sáez, *Phys. Rev.* **D32**, 2575 (1985).
- [97] K. Izumi, J.-A. Gu, and Y. C. Ong, *Phys. Rev.* **D89**, 084025 (2014), [arXiv:1309.6461 \[gr-qc\]](#) .
- [98] R.-J. Yang, *Europhys. Lett.* **93**, 60001 (2011), [arXiv:1010.1376 \[gr-qc\]](#) .
- [99] Y. Kucukakca, *Eur. Phys. J.* **C74**, 3086 (2014), [arXiv:1407.1188 \[gr-qc\]](#) .
- [100] M. Hohmann *et al.*, *Phys. Rev.* **D88**, 084054 (2013), [arXiv:1309.0031 \[gr-qc\]](#) .
- [101] R. Epstein, and I. I. Shapiro, *Phys. Rev.* **D22**, 2947 (1980).
- [102] B. Li, T. P. Sotiriou, and J. D Barrow, *Phys. Rev.* **D83**, 064035 (2011), [arXiv:1010.1041 \[gr-qc\]](#) .
- [103] J. Hayward, *Phys. Rev.* **D20**, 3039 (1979).
- [104] J. Hayward, *Gen. Rel. Grav.* **13**, 43 (1981).